Recurrence Relations

Questions

Q1.

Solve the recurrence system

$$u_1 = 1 \qquad u_2 = 4$$
$$9u_{n+2} - 12u_{n+1} + 4u_n = 3n$$

(Total for question = 9 marks)

Q2.

A tree at the bottom of a garden needs to be reduced in height. The tree is known to increase in height by 15 centimetres each year.

On the first day of every year, the height is measured and the tree is immediately trimmed by 3% of this height.

When the tree is measured, before trimming on the first day of year 1, the height is 6 metres.

Let H_n be the height of the tree immediately before trimming on the first day of year n.

(a) Explain, in the context of the problem, why the height of the tree may be modelled by the recurrence relation

$$H_{n+1} = 0.97H_n + 0.15, \quad H_1 = 6, \quad n \in \mathbb{Z}^+$$

(3)

(b) Prove by induction that $H_n = 0.97^{n-1} + 5$, $n \geq 1$

(4)

(c) Explain what will happen to the height of the tree immediately before trimming in the long term.

(1)

(d) By what fixed percentage should the tree be trimmed each year if the height of the tree immediately before trimming is to be 4 metres in the long term?

(2)

(Total for question = 10 marks)

Q3.

A recurrence system is defined by

$$u_{n+2} = 9(n+1)^2 u_n - 3u_{n+1} \qquad n \geqslant 1$$

$$u_1 = -3,\ u_2 = 18$$

Prove by induction that, for $n \in \mathbb{N}$,

$$u_n = (-3)^n n!$$

(6)

(Total for question = 6 marks)

Q4.

On Jim's 11th birthday his parents invest £1000 for him in a savings account.

The account earns 2% interest each year.

On each subsequent birthday, Jim's parents add another £500 to this savings account.

Let U_n be the amount of money that Jim has in his savings account n years after his 11th birthday, once the interest for the previous year has been paid and the £500 has been added.

(a) Explain, in the context of the problem, why the amount of money that Jim has in his savings account can be modelled by the recurrence relation of the form

$$U_n = 1.02U_{n-1} + 500 \qquad U_0 = 1000 \qquad n \in \mathbb{Z}^+$$

(3)

(b) State an assumption that must be made for this model to be valid.

(1)

(c) Solve the recurrence relation

$$U_n = 1.02U_{n-1} + 500 \qquad U_0 = 1000 \qquad n \in \mathbb{Z}^+$$

(5)

Jim hopes to be able to buy a car on his 18th birthday.

(d) Use the answer to part (c) to find out whether Jim will have enough money in his savings account to buy a car that costs £4 500

(2)

(Total for question = 11 marks)

Q5.

A population of deer on a large estate is assumed to increase by 10% during each year due to natural causes.

The population is controlled by removing a constant number, Q, of the deer from the estate at the end of each year.

At the start of the first year there are 5000 deer on the estate.

Let P_n be the population of deer at the end of year n.

(a) Explain, in the context of the problem, the reason that the deer population is modelled by the recurrence relation

$$P_n = 1.1P_{n-1} - Q, \qquad P_0 = 5000, \qquad n \in \mathbb{Z}^+$$

(3)

(b) Prove by induction that $P_n = (1.1)^n (5000 - 10Q) + 10Q, \quad n \geq 0$

(5)

(c) Explain how the long term behaviour of this population varies for different values of Q.

(2)

(Total for question = 10 marks)

Q6.

The number of visits to a website, in any particular month, is modelled as the number of visits received in the previous month plus k times the number of visits received in the month before that, where k is a positive constant.

Given that V_n is the number of visits to the website in month n,

(a) write down a general recurrence relation for V_{n+2} in terms of V_{n+1}, V_n and k.

(1)

For a particular website you are given that

- $k = 0.24$
- In month 1, there were 65 visits to the website.
- In month 2, there were 71 visits to the website.

(b) Show that

$$V_n = 50(1.2)^n - 25(-0.2)^n$$

(5)

This model predicts that the number of visits to this website will exceed one million for the first time in month N.

(c) Find the value of N.

(2)

(Total for question = 8 marks)

Q7.

A staircase has n steps. A tourist moves from the bottom (step zero) to the top (step n). At each move up the staircase she can go up either one step or two steps, and her overall climb up the staircase is a combination of such moves.

If u_n is the number of ways that the tourist can climb up a staircase with n steps,

(a) explain why u_n satisfies the recurrence relation

$$u_n = u_{n-1} + u_{n-2}, \text{ with } u_1 = 1 \text{ and } u_2 = 2$$

(3)

(b) Find the number of ways in which she can climb up a staircase when there are eight steps.

(1)

A staircase at a certain tourist attraction has 400 steps.

(c) Show that the number of ways in which she could climb up to the top of this staircase is given by

$$\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{401} - \left(\frac{1-\sqrt{5}}{2}\right)^{401}\right]$$

(5)

(Total for question = 9 marks)

Mark Scheme – Recurrence Relations

Q1.

Question	Scheme	Marks	AOs
	Auxiliary equation is $9r^2 - 12r + 4 = 0$, so $r = ...$	M1	1.1b
	$(3r-2)^2 = 0 \Rightarrow r = \frac{2}{3}$ is repeated root.	A1	1.1b
	Complementary function is $x_n = (A+Bn)\left(\frac{2}{3}\right)^n$ or $A\left(\frac{2}{3}\right)^n + Bn\left(\frac{2}{3}\right)^n$	M1	2.2a
	Try particular solution $y_n = an + b \Rightarrow 9(a(n+2)+b) - 12(a(n+1)+b) + 4(an+b) = 3n$	M1	2.1
	$\Rightarrow an + 6a + b = 3n \Rightarrow a = ..., b = ...$	dM1	1.1b
	$a = 3, b = -18$	A1	1.1b
	General solution is $u_n = x_n + y_n = (A+Bn)\left(\frac{2}{3}\right)^n + 3n - 18$	B1ft	2.2a
	$\left.\begin{array}{l} u_1 = 1 \Rightarrow 1 = \left(\frac{2}{3}\right)(A+B) - 15 \\ u_2 = 4 \Rightarrow 4 = \left(\frac{4}{9}\right)(A+2B) - 12 \end{array}\right\} A = ..., B = ...$	M1	2.1
	$u_n = 12(n+1)\left(\frac{2}{3}\right)^n + 3n - 18$ oe	A1	1.1b
		(9)	
			(9 marks)

Notes:

M1: Forms and solves the auxiliary equation.
A1: Correct (repeated) root found.
M1: Forms the correct complementary function for their (real) root(s) to the equation, $(A+Bn)r^n$ if repeated root, or allow $Ar_1^n + Br_2^n$ if distinct real roots are found.
M1: Attempts to use a particular solution of the correct form (ie $an + b$ or a higher order polynomial in n containing this) in the recurrence relation.
dM1: Expands and solves for a and b
A1: Correct values for a and b
B1ft: Forms the general solution as the sum of their complementary function and a particular solution of correct form with their a and b
M1: Applies the initial values and solves for the constants
A1: Correct answer.

Q2.

Question	Scheme	Marks	AOs
(a)	H_n is the measured height at the start of year n and this is decreased by 3% at the start of year n, so is multiplied by 97% = 0.97 to give 0.97 H_n as the new height due to trimming	B1	3.3
	0.15 is added to 0.97 H_n as 0.15 is 15 cm in m and this is how much the tree grows in a year.	B1	3.4
	And $H_1 = 6$ is the height of the tree at the start of year 1 before trimming	B1	1.1b
		(3)	
(b)	$n=1 \Rightarrow H_1 = (0.97)^{1-1} + 5 = 6$ So true for $n = 1$	B1	2.1
	Assume true for $n = k$ so $H_k = (0.97)^{k-1} + 5$ so $H_{k+1} = 0.97\left((0.97)^{k-1} + 5\right) + 0.15$	M1	2.4
	so $H_{k+1} = (0.97)^k + 4.85 + 0.15 = (0.97)^k + 5$	A1	1.1b
	If true for $n = k$ then true for $n = k + 1$, true for $n = 1$ so true for all (positive integers) n (Allow "for all values")	B1	2.2a
		(4)	
(c)	The height will approach 5m	B1	1.1b
		(1)	
(d)	Require $4 = 4x + 0.15$	M1	3.1b
	$x = 0.9625$ so 3.75%	A1	1.1b
		(2)	
			(10 marks)

Notes
(a) B1: Need to see 3% decrease linked to scale factor of 0.97 B1: Need to see that adding 0.15 corresponds to the yearly growth **in metres**. There must be some reference to the units for this mark. B1: An explanation that H_1 is the first term (the starting height) and this is 6m (b) B1: Begins proof by induction by considering $n = 1$ and obtains $H_1 = 6$ M1: Assumes true for $n = k$ and uses iterative formula to consider $n = k + 1$ A1: Reaches $(0.97)^k + 5$ with no errors B1: Correct conclusion. This mark is dependent on all previous marks apart from the first B mark. It is gained by conveying the ideas of **all** four underlined points **either** at the end of their solution **or** as a narrative in their solution. (c) B1: States the height will approach 5m (d) M1: Uses the model to adopt a correct strategy to find the required percentage A1: Interprets their answer correctly in terms of the original context

Q3.

Question	Scheme	Marks	AOs
	$n=1: u_1=(-3)^1\times1!=-3$ $n=2: u_2=(-3)^2\times2!=9\times2=18$ Hence true for $n=1$ and $n=2$	B1	2.2a
	Assume true for some $n=k$ and $n=k+1$, so $u_k=(-3)^k k!$ and $u_{k+1}=(-3)^{k+1}(k+1)!$	M1	2.4
	Then $u_{k+2}=9(k+1)^2\left((-3)^k k!\right)-3\left((-3)^{k+1}(k+1)!\right)$	M1	1.1b
	$=(-3)^k k!\left[9(k+1)^2-3(-3)(k+1)\right]$	M1	1.1b
	$=(-3)^k k!\left[9(k+1)(k+1+1)\right]=(-3)^k\times(-3)^2\times(k+1)(k+2)k!$ $=(-3)^{k+2}(k+2)!$	A1	2.1
	Hence if true for $n=k$ and $n=k+1$ then true for $n=k+2$. As also true for $n=1$ and $n=2$, then true for all $n\in\mathbb{N}$ by mathematical induction.	A1	2.4
		(6)	
		(6 marks)	

B1: Checks the closed form works for $n=1$ and $n=2$

M1: Makes the inductive assumption. May use e.g. $n=k-2$ and $n=k-1$ instead and show true for $n=k$. It must be clear it is the closed forms they are assuming, not a recurrence form.

M1: Substitutes expression for $n=k$ and $n=k+1$ (or equivalents) into the recurrence formula.

M1: Takes out common factors of at least $(-3)^k k!$ in their expression, or equivalent for their assumed true values. Treatment of the (-3) must be correct, but condone invisible brackets if recovered.

Note: they may well take out more at this stage, which is fine, e.g.
$u_{k+2}=9(k+1)^2\left((-3)^k k!\right)-3\left((-3)^{k+1}(k+1)!\right)=(-3)^{k+2}(k+1)!\left[(k+1)+1\right]$

A1: Simplifies correctly to the required form for their assumed true values.

A1: Correct conclusion made. Depends on all three M's and the A being gained. Must convey the ideas of 1) true for $n=1$ and $n=2$, 2) if true for two successive cases, it is also true for the next case and 3) a suitable conclusion that it is true for all positive n.

Q4.

Question	Scheme	Marks	AOs
(a)	U_{n-1} is the amount in the saving account $n-1$ years after Jim's 11th birthday. This is increased by 2% each year, so is multiplied by 1.02 to give $1.02U_{n-1}$	B1	3.3
	Jim's parents invest £500 for each subsequent birthday so 500 is added	B1	3.4
	$U_0 = 1000$ as this is the amount invested on Jim's 11th birthday	B1	1.1b
		(3)	
(b)	To use this model, one of, for example The interest rate stays the same each year Jim does not withdraw any money from the savings account Jim only saves the birthday money +£500 in this saving account, he does not invest any other money.	B1	3.5b
		(1)	
(c)	A complete method to solve the recurrence relation using $U_n = \text{CF} + \text{PS} = c(1.02)^n + \lambda$	M1	3.1a
	$\text{PS} = \lambda \quad \Rightarrow \lambda = 1.02\lambda + 500$ leading to $\lambda = \ldots$	M1	1.1b
	$\lambda = -25\,000$	A1	1.1b
	Uses $U_0 = 1000$ and their value for λ to find the value of $1000 = c(1.02)^0 - 25\ 000$ $c = \ldots(26\ 000)$	M1	1.1b
	$U_n = 26\ 000(1.02)^n - 25\ 000 \qquad (n \geq 0)$	A1	1.1b
		(5)	

	Alternative 1 Realises that $U_n =$ term of a GP + sum of a GP both with $r = 1.02$	M1	3.1a
	Sum of a GP $= \dfrac{500(1-1.02^n)}{1-1.02}$ or $\dfrac{500(1.02^n-1)}{1.02-1}$	M1 A1	1.1b 1.1b
	Term of a GP $= 1000(1.02)^n$ or $1000(1.02)^{n-1}$	M1	1.1b
	$U_n = 1000(1.02)^n - 25\ 000(1-1.02^n)$ or $U_n = 1000(1.02)^n + 25\ 000(1.02^n - 1)$	A1	1.1b
		(5)	
(d)	Uses $U_n = 26\ 000(1.02)^n - 25\ 000$, with either $n = 7$ or 8	M1	3.4
	$U_7 = 4865.83 > 4500$ therefore, Jim will have enough money in his savings account to buy a car costing £ 4500.	A1ft	2.2a
		(2)	
			(11 marks)

Notes
(a) **B1:** Need to explain that 2% interest rate linked to multiplication by scale factor 1.02 **B1:** Need to explain that 500 is added due to receiving £500 each year **B1:** Needs to explain that $U_0 = 1000$ is the initial amount invested
(b) **B1:** See main scheme
(c) **M1:** A complete method to solve the recurrence relation using $U_n = \text{CF} + \text{PS} = c(1.02)^n + \lambda$ **M1:** Uses $\text{PS} = \lambda \quad \Rightarrow \lambda = 1.02\lambda + 500$ to find a value for λ **A1:** $\lambda = -25\ 000$ **M1:** Uses U_0 and their value for λ to find a value of c **A1:** Fully correctly defined sequence $U_n = 26000(1.02)^n - 25\ 000, \quad (n \geq 0)$
Alternative 1 **M1:** A correct form for U_n term of a GP + Sum of a GP both with $r = 1.02$ **M1:** For the sum of a GP with $a = 500$, $r = 1.02$ and uses n or $n-1$ **A1:** Correct the sum of a GP with $a = 500$, $r = 1.02$ and n **M1:** For the term of a GP with $a = 1000$, $r = 1.02$ and uses n or $n-1$ **A1:** Fully correctly defined sequence U_n
(d) **M1:** Uses their U_n with either $n = 7$ or 8 **A1ft:** Finds U_7 compares with 4 500 and comes to an appropriate conclusion. Follow through on their value of U_7

Q5.

Question	Scheme	Marks	AOs
(a)	P_{n-1} is the population at the end of year $n-1$ and this is increased by 10% by the end of year n, so is multiplied by 110% = 1.1 to give $1.1 \times P_{n-1}$ as new population by natural causes	B1	3.3
	Q is subtracted from $1.1 \times P_{n-1}$ as Q is the number of deer removed from the estate	B1	3.4
	So $P_n = 1.1P_{n-1} - Q$, $P_0 = 5000$ as population at start is 5000 and $n \in Z^+$	B1	1.1b
		(3)	
(b)	Let $n = 0$, then $P_0 = (5000 - 10Q)(1.1)^0 + 10Q = 5000$ so result is true when $n = 0$	B1	2.1
	Assume result is true for $n = k$, $P_k = (1.1)^k(5000 - 10Q) + 10Q$, then as $P_{k+1} = 1.1P_k - Q$, so $P_{k+1} = \ldots$	M1	2.4
	$P_{k+1} = 1.1 \times 1.1^k(5000 - 10Q) + 1.1 \times 10Q - Q$	A1	1.1b
	So $P_{k+1} = (5000 - 10Q)(1.1)^{k+1} + 10Q$,	A1	1.1b
	Implies result holds for $n = k + 1$ and so by induction $P_n = (5000 - 10Q)(1.1)^n + 10Q$, is true for all integer n	B1	2.2a
		(5)	
(c)	For $Q < 500$ the population of deer will grow, for $Q > 500$ the population of deer will fall	B1	3.4
	For $Q = 500$ the population of deer remains steady at 5000,	B1	3.4
		(2)	
			(10 marks)

Notes:

(a)
B1: Need to see 10% increase linked to multiplication by scale factor 1.1
B1: Needs to explain that subtraction of Q indicates the removal of Q deer from population
B1: Needs complete explanation with mention of $P_n = 1.1P_{n-1} - Q$, $P_0 = 5000$ being the initial number of deer

(b)
B1: Begins proof by induction by considering $n = 0$
M1: Assumes result is true for $n = k$ and uses iterative formula to consider $n = k + 1$
A1: Correct algebraic statement
B1: Correct statement for $k + 1$ in required form
B1: Completes the inductive argument

(c)
B1: Consideration of both possible ranges of values for Q as listed in the scheme
B1: Gives the condition for the steady state

Q6.

Question	Scheme	Marks	AOs
(a)	$V_{n+2} = V_{n+1} + kV_n$	B1	3.3
		(1)	
(b)	$\lambda^2 - \lambda - 0.24 = 0 \Rightarrow \lambda = ...(1.2, -0.2)$	M1	1.1b
	$V_n = a(1.2)^n + b(-0.2)^n$	A1	2.2a
	$65 = a(1.2)^1 + b(-0.2)^1$ **and** $71 = a(1.2)^2 + b(-0.2)^2$	B1ft	3.4
	E.g. $\left.\begin{matrix} 78 = 1.44a - 0.24b \\ 71 = 1.44a + 0.04b \end{matrix}\right\} \Rightarrow 7 = -0.28b \Rightarrow b = ...$	M1	2.1
	$a = 50, b = -25 \Rightarrow V_n = 50(1.2)^n - 25(-0.2)^n$ *	A1*	1.1b
		(5)	
(c)	$50(1.2)^N > 10^6 \Rightarrow N = ...$	M1	3.1b
	$\Rightarrow N = 55$ i.e. month 55	A1	3.2a
		(2)	
			(8 marks)

Notes
(a) B1: A correct expression for the model using the information given (b) M1: Forms and solves the auxiliary equation for their answer to (a) with $k = 0.24$ A1: The **correct** closed form deduced from their solutions. This must be consistent with their equation. Note the answer is given so check carefully. This is **not** a follow through mark. B1ft: Applies initial conditions to their general equation – correct two equations for their general form with $V_1 = 65$ and $V_2 = 71$ M1: Attempts to solve their equations showing a correct method, reaching a value for at least one variable. It is a show that question and answers are on the paper, so method is needed. Look for one equation multiplied through to give same coefficients before attempting eliminating or substitution. If a matrix system is used the inverse must be found, not just solutions stated. A1*: Correct expression formed following suitable working with no errors seen. With fractions instead of decimals is fine. (c) M1: Selects a suitable method to solve the problem. For example, realises that in the model, $(-0.2)^n$ is negligible for large n and so attempts to solve e.g. $50(1.2)^N = 10^6$, or tries at least one value either side of $N = 55$ as a process of trial and improvement, or uses a calculator/graphical approach – implied by a value of $N = 55$ or $N = 54$ stated. A1: $N = 55$. The correct answer will imply both marks for this part. Ignore erroneous working if correct answer is stated as a restart.

(b) **Alt**	$V_n = 50(1.2)^n - 25(-0.2)^n,\ V_1 = 65,\ V_2 = 71$ $V_1 = 50 \times 1.2 - 25 \times -0.2 = 60 + 5 = 65$ $V_2 = 50 \times (1.2)^2 - 25 \times (-0.2)^2 = 72 - 1 = 71$	M1	1.1b
	Hence true for $n = 1$ and $n = 2$ Assume true for $n = k$ and $n = k + 1$ (for some $k > 0$)	A1	2.2a
	$V_{k+2} = V_{k+1} + 0.24V_k$ $= 50(1.2)^{k+1} - 25(-0.2)^{k+1} + 0.24\left(50(1.2)^k - 25(-0.2)^k\right)$	B1ft	3.4
	$= \frac{50}{1.2}(1.2)^{k+2} - \frac{25}{-0.2}(-0.2)^{k+2} + \frac{12}{1.2^2}(1.2)^{k+2} - \frac{6}{(-0.2)^2}(-0.2)^{k+2}$ $= \frac{125}{3}(1.2)^{k+2} + 125(-0.2)^{k+2} + \frac{25}{3}(1.2)^{k+2} - 150(-0.2)^{k+2} = \ldots$	M1	2.1
	So $V_{k+2} = 50(1.2)^{k+2} - 25(-0.2)^{k+2}$ Hence true for $n = k + 2$. So the result is true for $n = 1$ and $n = 2$, and if true for $n = k$ and $n = k + 1$ then it is true for $n = k + 2$. Hence by mathematical induction, for all $n \in \square$ $V_n = 50(1.2)^n - 25(-0.2)^n$ *	A1*	1.1b
		(5)	

Notes
M1: Substitutes into equation for $n = 1$ and $n = 2$ to verify true for these cases. A1: Deduces true for base cases and makes a **correct** assumption statement. This must include two successive cases assumed true, so e.g. as in scheme, or with $k - 2$ and $k - 1$ etc, or may assume true for all (integers) $k \leq n$. But do not allow if assumed true for just k. B1ft: Substitutes the formula for k and $k + 1$ (or their successive values) into the recurrence formula, follow through their equation from part (a). M1: Rearranges to the form $a(1.2)^{(k+2)} + b(-0.2)^{k+2}$ A1*: Correct work leading to the correct equation for V_{k+2} and makes suitable inductive conclusion, including the ideas of "true for $n = 1$ and $n = 2$", "if true for $n = k$ and $n = k + 1$ then true for $n = k + 2$" and "hence true for all integers".

Q7.

Question	Scheme	Marks	AOs
(a)	$u_1 = 1$ as there is only one way to go up one step	B1	2.4
	$u_2 = 2$ as there are two ways: one step then one step or two steps	B1	2.4
	If first move is one step then can climb the other $(n-1)$ steps in u_{n-1} ways If first move is two steps can climb the other $(n-2)$ steps in u_{n-2} ways So $u_n = u_{n-1} + u_{n-2}$	B1	2.4
		(3)	
(b)	Sequence begins 1, 2, 3, 5, 8, 13, 21, 34,… so 34 ways of climbing 8 steps	B1	1.1b
		(1)	
(c)	To find general term use $u_n = u_{n-1} + u_{n-2}$ gives $\lambda^2 = \lambda + 1$	M1	2.1
	This has roots $\frac{1 \pm \sqrt{5}}{2}$	A1	1.1b
	So general form is $A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n$	M1	2.2a
	Uses initial conditions to find A and B reaching two equations in A and B	M1	1.1b
	Obtains $A = \left(\frac{1+\sqrt{5}}{2\sqrt{5}}\right)$ and $B = -\left(\frac{1-\sqrt{5}}{2\sqrt{5}}\right)$ and so when $n = 400$ obtains $\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{401} - \left(\frac{1-\sqrt{5}}{2}\right)^{401}\right]$ *	A1*	1.1b
		(5)	
			(9 marks)

Notes:	
(a)	
B1:	Need to see explanation for $u_1 = 1$
B1:	Need to see explanation for $u_2 = 2$ with the two ways spelled out
B1:	Need to see the first move can be one step or can be two steps and clear explanation of the Iterative expression as in the scheme
(b)	
B1:	The answer is enough for this mark

Notes: (continued)	
(c)	
M1:	Obtains this characteristic equation
A1:	Solves quadratic – giving exact answers
M1:	Obtains a general form
M1:	Use initial conditions to obtains two equations which should be $A(1+\sqrt{5}) + B(1-\sqrt{5}) = 2$ o.e. and $A(3+\sqrt{5}) + B(3-\sqrt{5}) = 4$ but allow slips here.
A1*:	Must see exact correct values for A and B and conclusion given for $n = 400$